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Spectral Properties of Kirkwood–Salsburg and Kirkwood–Ruelle Operators

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A method for solving Kirkwood-type equations in Banach spaces $E_\xi(\Lambda)$ and $E_\xi^s(\Lambda)$ is applied to derive spectral properties of Kirkwood–Salsburg and Kirkwood–Ruelle operators in these spaces. For stable interactions these operators are shown to have, besides the point spectrum, a residual one. We establish also that the residual spectrum may disappear if a superstable (singular) interaction between particles is switched on. In this case the bounded Kirkwood–Salsburg operator is proved to have a zero Fredholm radius.

KEY WORDS: Kirkwood–Salsburg and Kirkwood–Ruelle operators; spectrum; Fredholm radius.

1. INTRODUCTION

In a previous paper[1] we have derived the Kirkwood–Salsburg (KS), Kirkwood–Ruelle (KR), and Mayer–Montroll equations for classical continuous systems in a finite volume for nonempty boundary conditions. Then, by using an analytic continuation in activity of the corresponding resolvents the uniqueness theorem for the solution is proved outside the well known analyticity circle in the activity plane, see Ref. 2, Section 4.2.

We now extend this program to study the spectral properties of the KS and KR operators for tempered boundary conditions in Banach spaces $E_\xi(\Lambda)$ and $E_\xi^s(\Lambda)$. The main part of this paper will, in fact, consist in obtaining the structure of the spectrum and showing how it depends on the particle interaction and the choice of the operator domain. We prove that in a general case of stable interactions the KS and KR operators defined in a bounded region (finite volume) may have, besides the point spectrum, a

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residual one and generalized eigenvalues. The point spectrum and generalized eigenvalues coincide with the inverses of the zeros (in activity) of the grand canonical partition function for a fixed tempered configuration (boundary condition) outside the region. We also show that the residual spectrum may disappear when one goes to superstable interactions, e.g., singular pair potentials with a hard core or nonnegative potentials.

In this paper we will restrict ourselves mainly to the case of bounded regions. But in the instructive particular case of the ideal gas the spectrum of the KS operator is described also in the infinite-volume limit (thermodynamic limit).

We stick to the notation of Refs. 1 and 2 but for the reader's convenience recall very briefly some of the main definitions and results in Section 2. The main results and proofs are brought together in Section 3. Section 4 is devoted to discussion.

2. PRELIMINARIES

Let $\Lambda \subset \mathbb{R}^\nu$ be an open bounded simply connected region (container in a ν-dimensional space $\mathbb{R}^\nu$) with a finite volume $|\Lambda| = \operatorname{mes}\Lambda$. Let $\Omega = \{X\}$ refer to configurations of identical classical particles in $\mathbb{R}^\nu$ interacting by a (stable or superstable) pair potential $\Phi : \mathbb{R}^\nu \to \mathbb{R}^1 \cup \{+\infty\}$, which is a measurable function with $\Phi(x) = \Phi(-x)$; see Ref. 2. Let the restriction $\Omega \upharpoonright \Lambda = \Omega_\Lambda$ be the set of configurations in Λ. Then the subset $\Omega'_{\bar\Lambda} \subset \Omega_{\bar\Lambda}$, $\bar\Lambda = \mathbb{R}^\nu \backslash \Lambda$, is said to be the set of tempered configurations[1] if for any $X_{\bar\Lambda} \in \Omega'_{\bar\Lambda}$ there exists a finite $G(X_{\bar\Lambda}) \geqslant 0$ such that

$$W(X_\Lambda, X_{\bar\Lambda}) = \sum_{\substack{x \in X_\Lambda \\ y \in X_{\bar\Lambda}}} \Phi(x-y) \geqslant -G(X_{\bar\Lambda}) \operatorname{card} X_\Lambda \tag{2.1}$$

for arbitrary $X_\Lambda \in \Omega_\Lambda$. By definition $W(\phi, X_{\bar\Lambda}) = W(X_\Lambda, \phi) = 0$; $X_\Lambda = X_n = (x_1, \ldots, x_n)$ if $\operatorname{card} X_\Lambda = n$ and $X_{n=0} = \emptyset$. The grand canonical partition function then is of the form

$$\Xi(\beta, z, \Lambda \mid X_{\bar\Lambda}) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} dX_n \exp[-\beta U(X_n) - \beta W(X_n, X_{\bar\Lambda})] \tag{2.2}$$

The conditional correlation functions are defined as

$$\begin{aligned} \rho_\Lambda(z, X_m \mid X_{\bar\Lambda}) &= r_\Lambda(z, X_m \mid X_{\bar\Lambda})[\Xi(\beta, z, \Lambda \mid X_{\bar\Lambda})]^{-1} \\ r_\Lambda(z, X_m \mid X_{\bar\Lambda}) &= \chi_\Lambda(X_m) \sum_{n=0}^{\infty} \frac{z^{m+n}}{n!} \\ &\quad \times \int_{\Lambda^n} dY_n \exp[-\beta U(X_m \cup Y_n) - \beta W(X_m \cup Y_n, X_{\bar\Lambda})] \end{aligned} \tag{2.3}$$

Here $\chi_\Lambda(X_m)$ is the characteristic function of the set $\Lambda^m \subset \mathbb{R}^{m\nu}$, z is activity, β^{-1} is temperature, and $X_m \cup Y_n = (x_1, \ldots, x_m, y_1, \ldots, y_n)$. The convergence in (2.2) and (2.3) is ensured by the temperedness of the boundary condition: $X_{\bar\Lambda} \in \Omega'_{\bar\Lambda}$ and by the interaction stability[2]:

$$U(X_n) = \sum_{1 \leqslant i < j \leqslant n} \Phi(x_i - x_j) \geqslant -nB, \qquad B \geqslant 0 \tag{2.4}$$

Using (2.1)–(2.4) one gets in the physical region of activity $z \geqslant 0$ that

$$\rho_\Lambda(z, X_m \mid X_{\bar\Lambda}) \leqslant \chi_\Lambda(X_m) z^m e^{m\beta[G(X_{\bar\Lambda}) + B]} \times e^{z|\Lambda|\exp\{\beta[G(X_{\bar\Lambda}) + B]\}} \left[\Xi(\beta, z, \Lambda \mid X_{\bar\Lambda}) \right]^{-1} \tag{2.5}$$

Therefore, a powerful approach to study the correlation functions and solutions of correlation equations results from the observation[2-4] that $\{\rho_\Lambda(z, X_m \mid X_{\bar\Lambda})\}_{m \geqslant 1} = \rho_\Lambda(z \mid X_{\bar\Lambda})$ may be viewed as a vector of a vector space. This space consists of the sequences of complex-valued measurable functions $\{\varphi_m(X_m)\}_{m \geqslant 1} = \varphi$:

$$E_\xi = \left\{ \varphi : \sup_{m \geqslant 1} \xi^{-m} \|\varphi_m(X_m)\|_{L^\infty(\mathbb{R}^{m\nu})} = \|\varphi\|_\xi < \infty \right\} \tag{2.6}$$

and it is a Banach space with respect to $\|\cdot\|_\xi$-norm. Then, the vector $\rho_\Lambda(z \mid X_{\bar\Lambda})$ will belong to the subspace $E_\xi^s \subset E_\xi$ of the symmetric function sequences[1] for $\xi > z\exp[-\beta G(X_{\bar\Lambda}) - \beta B]$; see (2.5).

Now, let us define formally, on E_ξ, the linear operator K; see Ref. 2:

$$(K\varphi)(x_1) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{n\nu}} dY_n \, K_\Phi(x_1, Y_n) \varphi_n(Y_n)$$

$$(K\varphi)(X_m) = e^{-\beta W(x_1, X_m \setminus x_1)} \left\{ \varphi_{m-1}(X_m \setminus x_1) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{n\nu}} dY_n \times K_\Phi(x_1, Y_n) \varphi_{n+m-1}\left[Y_n \cup (X_m \setminus x_1) \right] \right\}, \quad m \geqslant 2 \tag{2.7}$$

$$K_\Phi(x_1, Y_n) = \prod_{y \in Y_n} \left[e^{-\beta\Phi(x_1 - y)} - 1 \right]$$

Then the KS operator for the container $\Lambda \subset \mathbb{R}^\nu$ with a tempered boundary condition is[1]

$$\begin{gathered} K_\Lambda = e^{-\beta \hat{W}_1(X_{\bar\Lambda})} \hat{\chi}_\Lambda K \\ (\hat{\chi}_\Lambda \varphi)(X_m) = \chi_\Lambda(X_m) \varphi_m(X_m) \\ \left[\hat{W}_1(X_{\bar\Lambda}) \hat{\chi}_\Lambda \varphi \right](X_m) = W(x_1, X_{\bar\Lambda}) \chi_\Lambda(X_m) \varphi_m(X_m) \end{gathered} \tag{2.8}$$

and the corresponding KS equation is

$$\varphi = z e^{-\beta \hat{W}_1(X_{\bar\Lambda})} \hat{\chi}_\Lambda \alpha + z K_\Lambda \varphi \tag{2.9}$$

where $\alpha = \{1, 0, 0, \ldots\}$ and $\varphi_0(\emptyset) = 1$; see (2.3). Analogously, we define the finite-volume KR operator:

$$\Pi K_\Lambda = e^{-\beta \hat{W}_1(X_{\bar{\Lambda}})} \hat{\chi}_\Lambda \Pi K \tag{2.10}$$

Here the "index-juggling" operator Π: $\varphi_m(X_m) \to \varphi_m(x_\pi \cup (X_m \backslash x_\pi))$ satisfies the condition (see Refs. 2–4)

$$W(x_\pi, X_m \backslash x_\pi) \geqslant -2B \tag{2.11}$$

Therefore, in contrast to the operator $K_\Lambda : E_\xi \to E_{e^{2\beta B}\xi}$ [see (2.8)], the KR operator (2.10) is bounded:

$$\|\Pi K_\Lambda\|_{E_\xi} \leqslant \xi^{-1} e^{\xi C(\beta)} e^{\beta[G(X_{\bar{\Lambda}}) + 2B]} \tag{2.12}$$

if the pair potential is regular in the sense of Ref. 2:

$$C(\beta) = \int_{\mathbb{R}^\nu} dx \, |e^{-\beta \Phi(x)} - 1| < \infty \tag{2.13}$$

Let $\varphi \in E_\xi^s$; then acting by the operator Π on both sides of Eq. (2.9) one gets the KR equation:

$$\varphi = z e^{-\beta \hat{W}_\pi(X_{\bar{\Lambda}})} \hat{\chi}_\Lambda \alpha + z \Pi K_\Lambda \varphi \tag{2.14}$$

The estimate (2.12) shows that if the domain $D(\Pi K_\Lambda)$ of the KR operator is taken as E_ξ, $E_\xi(\Lambda) = \hat{\chi}_\Lambda E_\xi$, or $E_\xi^s(\Lambda) = \hat{\chi}_\Lambda E_\xi^s$, the resolvent $R_\lambda(\Pi K_\Lambda) = (\lambda I - \Pi K_\Lambda)^{-1}$ is $\| \cdot \|_\xi$-analytic for

$$\lambda \in C_-(\xi) = \{\lambda \in \mathbb{C} : |\lambda| > \xi^{-1} e^{\xi C(\beta)} e^{\beta[G(X_{\bar{\Lambda}}) + 2B]}\} \tag{2.15}$$

Thus, $C_-(\xi) \subset P(\Pi K_\Lambda)$, where $P(\Pi K_\Lambda)$ is the resolvent set of the KR operator.

If $X_{\bar{\Lambda}} \in \Omega'_{\bar{\Lambda}}$ and the particle interaction is stable, the grand partition function (2.2) can be continued in the activity to an entire function of order at most one. Let $N'(\Xi) = \{z_i \in \mathbb{C} : \Xi(\beta, z_i, \Lambda \mid X_{\bar{\Lambda}}) = 0\}$; then for $z \notin N'(\Xi)$ one obtains an estimate similar to (2.5). Hence, in the region

$$C'_+(\xi) = \{z \in \mathbb{C} : |z| < \xi e^{-\beta[G(X_{\bar{\Lambda}}) + B]}\} \tag{2.16}$$

$\rho_\Lambda(z \mid X_{\bar{\Lambda}})$ is a vector-valued $\| \cdot \|_\xi$-meromorphic function [see (2.3)] with poles coinciding with the set $N'_+(\xi) = N'(\Xi) \cap C'_+(\xi)$. It is clear that for the activity in the region

$$C'_-(\xi) = \{z \in \mathbb{C} : z^{-1} \in C_-(\xi)\} \subset C'_+(\xi) \tag{2.17}$$

the solution of the KR equation (2.14) in the space $E_\xi^s(\Lambda)$ is unique and $\| \cdot \|_\xi$-analytic in z. Consequently, for the restriction $\Pi K_\Lambda \upharpoonright E_\xi^s(\Lambda)$ one gets

$$\varphi(z \mid X_{\bar{\Lambda}}) = R_{z^{-1}}(\Pi K_\Lambda) e^{-\beta \hat{W}_\pi(X_{\bar{\Lambda}})} \hat{\chi}_\Lambda \alpha \tag{2.18}$$

and the following proposition.

Proposition 2.1. (see Ref. 1, 13). Let $X_{\bar{\Lambda}} \in \Omega'_{\bar{\Lambda}}$ and the pair interaction potential be stable and regular. Then the KS and KR equations have in the space $E^s_\xi(\Lambda)$ the same unique solution (2.18) which is $\|\cdot\|_\xi$-analytic in the region $C'_-(\xi)$, $\|\cdot\|_\xi$-meromorphic in $C'_+(\xi)$, and coincides with $\rho_\Lambda(z \mid X_{\bar{\Lambda}}) \upharpoonright C'_+(\xi)$.

Remark 2.1. The set $N'(\Xi)$ does not depend on the choice of the parameter $\xi > 0$. Therefore, the maximal analyticity circle for the solution $\varphi(z \mid X_{\bar{\Lambda}})$ is

$$C'_0 = \left\{ z \in \mathbb{C} : |z| < a = \min_{i \geqslant 1} |z_i|,\ z_i \in N'(\Xi) \right\} \tag{2.19}$$

Then for $\xi > 0$ large enough the set $N'_+(\xi) \neq \{\emptyset\}$; see Fig. 1. The points represent there the set $N'(\Xi)$ and the region $\mathbb{C} \backslash C'_+(\xi)$ is hatched.

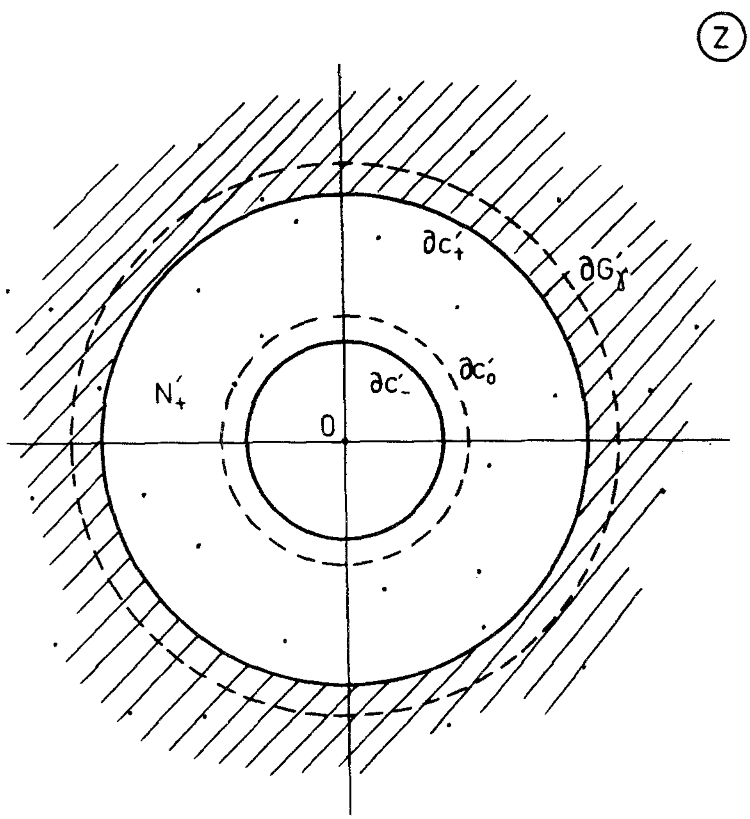

Fig. 1. Analytic properties of correlation vector-valued function $\rho_\Lambda(z \mid x_{\bar{\Lambda}})$ (see 2.18 and 3.7).

3. RESULTS

Ideal Gas

In this case the KS operator (2.8) has the following simple form [see (2.7)]:

$$K_\Lambda \varphi = \{0, \chi_\Lambda(X_2)\varphi_1(X_2 \backslash x_1), \ldots, \chi_\Lambda(X_m)\varphi_{m-1}(X_m \backslash x_1), \ldots\} \quad (3.1)$$

Then one has

$$\|K_\Lambda \varphi\|_\xi = \xi^{-1}\|\varphi\|_\xi, \qquad \varphi \in E_\xi(\in E_\xi(\Lambda)) \quad (3.2)$$

Therefore, for both the domains $D(K_\Lambda) = E_\xi$ and $D(K_\Lambda) = E_\xi(\Lambda)$ the spectrum $\sigma(\mathrm{K}_\Lambda) \subseteq C(\xi)$, where

$$C(\xi) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \xi^{-1}\} \quad (3.3)$$

Theorem 3.1. Let $\Phi(x) = 0$; then the spectrum $\sigma(K_\Lambda)$ of the KS operator with $D(K_\Lambda) = E_\xi(\Lambda)$ is residual and

$$\sigma(K_\Lambda) = \sigma_{\mathrm{res}}(K_\Lambda) = C(\xi) \quad (3.4)$$

Remark 3.1. The KS operator for the ideal gas (3.1) is very similar to the right-shift operator in l_∞. Thus, one can prove (3.4) by a proper modification of the arguments used for l_∞ in Ref. 5, Section VI.3. We propose below a line of reasoning that is more appropriate for our aim because it works also for nonideal systems; see Theorem 3.3.

Proof of Theorem 3.1. The spectrum of a linear operator, e.g., K_Λ, is known to be the union of three disjoint components: $\sigma(K_\Lambda) = \sigma_p(K_\Lambda) \cup \sigma_{\mathrm{cont}}(K_\Lambda) \cup \sigma_{\mathrm{res}}(K_\Lambda)$. Moreover, one gets $\sigma_{\mathrm{cont}}(K_\Lambda) = L(K_\Lambda)\backslash[\Gamma(K_\Lambda) \cup \sigma_p(K_\Lambda)]$ and $\sigma_{\mathrm{res}}(K_\Lambda) = \Gamma(K_\Lambda)\backslash\sigma_p(K_\Lambda)$, where $L(K_\Lambda)$ and $\Gamma(K_\Lambda)$ are, respectively, the limit and the compression spectra of the operator K_Λ.[2] As follows from (3.1), the equation

$$K_\Lambda \varphi = \lambda \varphi, \qquad \varphi \in E_\xi(\Lambda)$$

has only the trivial solution, i.e., the point spectrum $\sigma_p(K_\Lambda) = \{\varnothing\}$. Consider now the interior of the circle $C(\xi)$. From the KS equation (2.9) we find that the unique solution is of the form

$$\varphi(\lambda) = \{\lambda^{-m}\chi_\Lambda(X_m)\}_{m \geqslant 1} = \rho_\Lambda(z = \lambda^{-1}) \quad (3.5)$$

[2] See, e.g., P. Halmos, *A Hilbert Space Problem Book* (D. Van Nostrand Company, Princeton, New Jersey, 1967.) We use the name "limit spectrum" instead of the "approximate point spectrum" used in the Halmos book.

For $|\lambda| < \xi^{-1}$ one gets $\rho_\Lambda(\lambda^{-1}) \notin E_\xi(\Lambda)$. Hence the vector $\hat{\chi}_\Lambda \alpha \in E_\xi(\Lambda)$ does not belong to the range $\mathrm{Ran}(\lambda I - K_\Lambda)$. Note that for a fixed $\lambda \in C(\xi) \backslash \partial C(\xi)$ there exists a $\xi' > \xi$ such that $\rho_\Lambda(\lambda^{-1}) \in E_{\xi'}(\Lambda)$, see (3.5). Then, by the open mapping theorem $(\lambda I - K_\Lambda): \mathcal{U}_{\xi'}(\rho_\Lambda(\lambda^{-1})) \to \mathcal{U}_{\xi'}(\hat{\chi}_\Lambda \alpha)$, where $\mathcal{U}_{\xi'}(\cdot)$ is a neighborhood of the corresponding vector in $E_{\xi'}(\Lambda)$. A little reflection shows that the imbedding $E_\xi(\Lambda) \subset E_{\xi'}(\Lambda)$ of Banach spaces for $\xi' > \xi$ implies (i) the subspace $E_\xi(\Lambda)$ is not dense in the space $E_{\xi'}(\Lambda)$ in the topology defined by the norm $\|\cdot\|_{\xi'}$; (ii) if $\varphi \in E_\xi(\Lambda)$, then $\mathcal{U}_{\xi'}(\varphi) \cap E_\xi(\Lambda) = \mathcal{U}_\xi(\varphi)$ is a neighborhood of the vector φ in $E_\xi(\Lambda)$. Therefore, in the space $E_{\xi'}(\Lambda)$ there is a neighborhood $\mathcal{I}_{\xi'}(\rho_\Lambda(\lambda^{-1}))$ such that $\mathcal{I}_{\xi'}(\rho_\Lambda(\lambda^{-1})) \cap E_\xi(\Lambda) = \{\emptyset\}$. Together with $\sigma_p(K_\Lambda) = \{\emptyset\}$ this means that the $\mathrm{Ran}(\lambda I - K_\Lambda)$ does not intersect the neighborhood $\mathcal{I}_\xi(\hat{\chi}_\Lambda \alpha) = \{(\lambda I - K_\Lambda)\mathcal{I}_{\xi'}(\rho_\Lambda(\lambda^{-1}))\} \cap E_\xi(\Lambda)$. Hence we get $C(\xi)\backslash \partial C(\xi) \subset \Gamma(K_\Lambda)$. It remains to consider the boundary $\partial C(\xi)$. Using (3.2) one gets

$$\|(\lambda I - K_\Lambda)\varphi\|_\xi \geqslant \big||\lambda| - \xi^{-1}\big| \, \|\varphi\|_\xi$$

Then $L(K_\Lambda) = \partial C(\xi)$ because always the boundary $\partial\sigma(K_\Lambda) \subset L(K_\Lambda)$. On the other hand, for arbitrary $\lambda \in \partial C(\xi)$ consider the vector $\rho_\Lambda(\lambda^{-1}) \in E_\xi(\Lambda)$; see (3.5). The equation

$$\rho_\Lambda(\lambda^{-1}) = (\lambda I - K_\Lambda)\psi$$

has the unique solution

$$\psi[\rho_\Lambda] = \left\{ \lambda^{-(m+1)} \chi_\Lambda(X_m) \sum_{n=0}^{m} \lambda^n \rho_\Lambda(\lambda^{-1}, X_n) \right\}_{m \geqslant 1} \notin E_\xi(\Lambda)$$

Thus, the vector $\rho_\Lambda(\lambda^{-1}) \notin \mathrm{Ran}(\lambda I - K_\Lambda)$ for $|\lambda| = \xi^{-1}$. Suppose now that the vector $\mu \in \mathcal{B}_{\xi,\epsilon}(\rho_\Lambda(\lambda^{-1})) = \{\varphi : \|\varphi - \rho_\Lambda(\lambda^{-1})\|_\xi < \epsilon\}$ for some $\epsilon > 0$, then we have

$$|\mathrm{Re}\,\lambda^n \mu_n(X_n) - \mathrm{Re}\,\lambda^n \rho_\Lambda(\lambda^{-1}, X_n)| \leqslant \xi^{-n} |\mu_n(X_n) - \rho_\Lambda(\lambda^{-1}, X_n)| < \epsilon$$

and, consequently, $\mathrm{Re}\,\lambda^n \mu_n(X_n) > 1 - \epsilon$. Therefore, one gets $|\psi_m[\mu](X_m)| \geqslant m(1-\epsilon)\xi^{m+1}$. Hence, the ball $\mathcal{B}_{\xi,\epsilon}(\rho_\Lambda(\lambda^{-1}))$ is also disjoint from the $\mathrm{Ran}(\lambda I - K_\Lambda)$, i.e., $\partial C(\xi) \subset \Gamma(K_\Lambda)$ and the continuous spectrum $\sigma_{\mathrm{cont}}(K_\Lambda) = \{\emptyset\}$. The collection of the above results and the definition of $\sigma_{\mathrm{res}}(K_\Lambda)$ complete the proof. ■

Corollary 3.1 (Thermodynamic limit). The KS operator for the ideal gas in the limit of the infinite container has the form $K = K_{\Lambda = \mathbb{R}^\nu}$ and $\|K\|_{E_\xi} = \|K_\Lambda\|_{E_\xi(\Lambda)}$; see (2.7) and (3.1). The above line of reasoning gives that the spectrum $\sigma(K)$ for $D(K) = E_\xi$ is also residual and

$$\sigma(K) = \sigma_{\mathrm{res}}(K) = \sigma_{\mathrm{res}}(K_\Lambda \upharpoonright E_\xi(\Lambda)) \tag{3.6}$$

see Fig. 2.

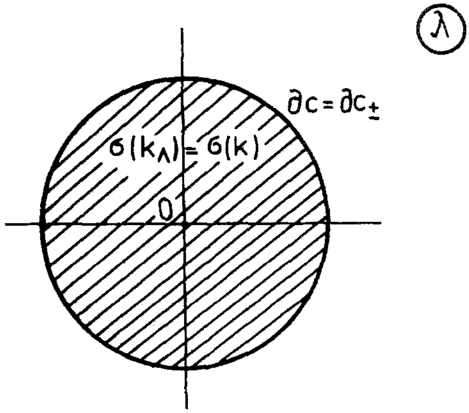

Fig. 2. Spectrum of KS operator for ideal gas in finite- and infinite-volume cases. Spectrum is purely residual.

Remark 3.2. Results of Theorem 3.1 remain unchanged if one considers the restriction $K_\Lambda \upharpoonright E_\xi^s(\Lambda)$. For the extension of K_Λ to $D(K_\Lambda) = E_\xi$ (or $= E_\xi^s$) one gets $\operatorname{Ker} K_\Lambda \neq \{\varphi = 0\}$ and, as a result, $L(K_\Lambda) \cap \Gamma(K_\Lambda) = \{\lambda = 0\}$.

Stable Interactions

Now the KS operator (2.8) is not bounded, in general, in any of the spaces $E_\xi, E_\xi(\Lambda), E_\xi^s(\Lambda)$; see Section 2. Therefore, it is relevant here to consider also the KR operator (2.10). We shall start with spectral properties of the KS operator.

Theorem 3.2. Let $X_{\bar{\Lambda}} \in \Omega'_{\bar{\Lambda}}$ and the pair potential $\Phi(x) \neq 0$ be stable and regular. Then one gets

(i) for $D(K_\Lambda) = E_\xi$ the spectrum $\sigma_p(K_\Lambda) \neq \{\varnothing\}$, if $\xi > ae^{\beta[G(X_{\bar{\Lambda}}) + B]}$ and $\{\lambda = 0\} \in L(K_\Lambda) \cap \Gamma(K_\Lambda)$;

(ii) for $D(K_\Lambda) = E_\xi^s(\Lambda)$ the spectrum $\sigma_p(K_\Lambda) \subset \{\lambda_i = z_i^{-1} : z_i \in N'(\Xi)\} = N(\Xi)$;

(iii) if there exists $\gamma > 0$ such that $r_\Lambda(z \mid X_{\bar{\Lambda}}) \notin E_\xi^s(\Lambda)$ for $z \in G'_\gamma(\xi) = \{z \in \mathbb{C} : |z| > \gamma\xi\}$ [see (2.16)], then for $D(K_\Lambda) = E_\xi^s(\Lambda)$ the set

$$\tilde{\sigma}_p(K_\Lambda) = N(\Xi) \cap G_\gamma(\xi) \tag{3.7}$$

corresponds to generalized eigenvalues; here $G_\gamma(\xi) = \{\lambda = z^{-1} : z \in G'_\gamma(\xi)\}$.

Proof. (i) The KS equation, as it originates,[2] is identically satisfied by the vector $r_\Lambda(z \mid X_{\bar{\Lambda}})$ if the free term has the form $z\Xi(\beta, z, \Lambda \mid X_{\bar{\Lambda}})$

$e^{-\beta \hat{W}_1(X_{\bar{\Lambda}})}\hat{\chi}_\Lambda \alpha$, see (2.3) and (2.9). Then the equation

$$K_\Lambda \varphi = \lambda \varphi, \qquad \varphi \in E_\xi \tag{3.8}$$

should (see Ref. 1 and Proposition 2.1) have nontrivial solutions $\{r_\Lambda(z_i \mid X_{\bar{\Lambda}})\}_{i \geqslant 1}$ for $\lambda_i = z_i^{-1}$, where $z_i \in N'(\Xi)$. Then the estimate (2.16) and Remark 2.1 give the first part of the desired result. The second part is the consequence of $\operatorname{Ker} K_\Lambda = \hat{\chi}_{\bar{\Lambda}} E_\xi$; compare with Remark 3.2.

(ii) In the subspace $E_\xi^s(\Lambda)$ there are no other solutions of Eq. (3.8) but $\{r_\Lambda(\lambda_i^{-1} \mid X_{\bar{\Lambda}})\}_{i \geqslant 1}$, $\lambda_i \in N(\Xi)$; see Ref. 1. Thus, we get for the point spectrum that $\sigma_p(K_\Lambda) \subset N(\Xi)$.

(iii) For $|z_i| > \gamma \xi$ we have $r_\Lambda(z_i \mid X_{\bar{\Lambda}}) \notin E_\xi^s(\Lambda)$. Hence the set (3.7) consists of generalized eigenvalues in the sense that corresponding eigenvectors $\{r_\Lambda(z_i \mid X_{\bar{\Lambda}})\}_{i \geqslant 1} \notin D(K_\Lambda)$. ■

Corollary 3.2. We recall that in (2.19) $a > 0$ and the set $N(\Xi)$ has the accumulation point at most at $\lambda = 0$. Then the estimate (2.16) and (iii) show that for $\xi > 0$ small enough we have $\sigma_p(K_\Lambda) = \{\varnothing\}$, i.e., the KS operator has only generalized eigenvalues.

Theorem 3.3. Let the assumptions of Theorem 3.2 be valid. Then for the KR operator (2.10) with domain $D(\Pi K_\Lambda) = E_\xi^s(\Lambda)$ one has

(i) $\sigma_p(\Pi K_\Lambda) \subset N(\Xi)$;

(ii) $\tilde{\sigma}_p(\Pi K_\Lambda) = N(\Xi) \cap G_\gamma(\xi)$ [see (3.7)];

(iii) $\sigma_{\mathrm{res}}(\Pi K_\Lambda) \supset G_\gamma(\xi) \backslash \tilde{\sigma}_p(\Pi K_\Lambda)$.

Proof. If one remarks that $\Pi \upharpoonright E_\xi^s = I$ and $\operatorname{Ker} \Pi = \{\varphi = 0\}$, then the proof of (i) and (ii) is an immediate consequence of the proof of (ii) and (iii) in Theorem 3.2.

(iii) Let $C_+(\xi) = \{\lambda = z^{-1} : z \in C'_+(\xi)\}$; then estimate (2.16) shows that for any $\lambda \in G_\gamma(\xi) \backslash N(\Xi)$ there is $\xi > \xi$ such that $\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}}) \in E_{\xi'}(\Lambda)$. Using Proposition 2.1 one deduces that for $\lambda \in G_\gamma(\xi) \backslash N(\Xi)$ the operator $(\lambda I - \Pi K_\Lambda)$ maps the vector $\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}}) \notin E_\xi(\Lambda)$ into the vector $\alpha_\Lambda = \exp[-\beta \hat{W}_\pi(X_{\bar{\Lambda}})]\hat{\chi}_\Lambda \alpha \in E_\xi(\Lambda)$. Because the operator $(\lambda I - \Pi K_\Lambda)$ is bounded (2.12), then by the open mapping theorem $(\lambda I - \Pi K_\Lambda)$: $\mathfrak{U}_{\xi'}(\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}})) \to \mathfrak{U}_{\xi'}(\alpha_\Lambda)$, where $\mathfrak{U}_{\xi'}(\cdot)$ denotes a neighborhood in $E_{\xi'}(\Lambda)$. Further, we can follow the arguments used in the proof of Theorem 3.1. Then, there is such a neighborhood $\mathcal{G}_{\xi'}(\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}}))$ that $\mathcal{G}_{\xi'}(\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}})) \cap E_\xi(\Lambda) = \{\varnothing\}$ for $\lambda \in G_\gamma(\xi) \backslash N(\Xi)$. Therefore, the set $\operatorname{Ran}(\lambda I - \Pi K_\Lambda)$ does not intersect neighborhood

$$\mathcal{G}_\xi(\alpha_\Lambda) = \left\{(\lambda I - \Pi K_\Lambda)\mathcal{G}_{\xi'}\left(\rho_\Lambda(\lambda^{-1} \mid X_{\bar{\Lambda}})\right)\right\} \cap E_\xi(\Lambda)$$

Hence, $\sigma_{\mathrm{res}}(\Pi K_\Lambda) \neq \{\varnothing\}$ and contains the set $G_\gamma(\xi) \backslash \tilde{\sigma}_p(\Pi K_\Lambda)$. ■

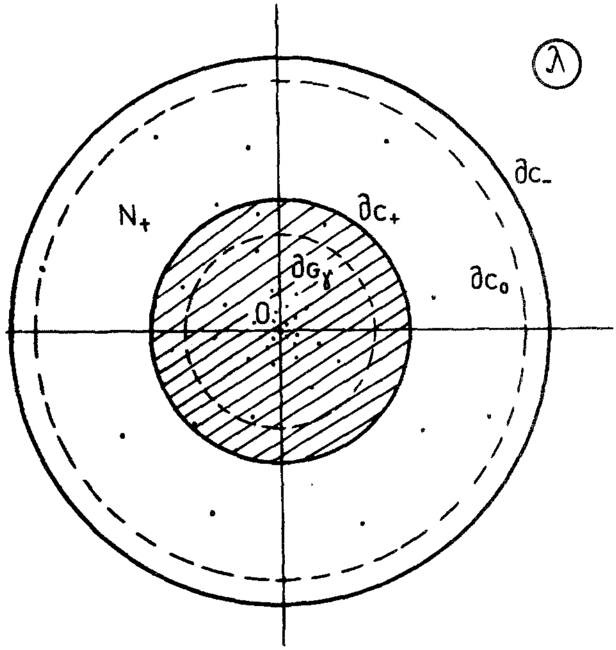

Fig. 3. General structure of KS(KR) operator spectrum.

Remark 3.3. The structure of $\sigma(\Pi K_\Lambda)$ is illustrated in Fig. 3. Here $C_0 = \{\lambda = z^{-1} : z \in C_0'\}$ [see (2.19)]; the hatched area $\mathbb{C}\backslash C_+(\xi)$ contains the circle $G_\gamma(\xi)$ and the set $N(\Xi)$ is marked by points. It should be stressed that the arguments in the proof of Theorems 3.2 and 3.3 do not eliminate the $\sigma_{\text{res}}(K_\Lambda)$ or $\sigma_{\text{cont}}(K_\Lambda)$ and $\sigma_{\text{res}}(\Pi K_\Lambda)$ or $\sigma_{\text{cont}}(\Pi K_\Lambda)$ in the circle $\mathbb{C}\backslash C_+(\xi)$.

Remark 3.4. The condition (iii) in Theorem 3.2 is clearly valid for the ideal gas; see (3.5). A more difficult problem is to single out a class of nontrivial pair potentials satisfying the condition (iii); see Ref. 1.

Theorem 3.4. Let a pair interaction potential be such that $\Phi(x) \geqslant 0$ and $\lim_{\epsilon\to 0} \|\Phi(x)\|_{L^\infty(x\in\mathbb{R}^\nu : |x|<\epsilon)} = 0$. Then condition (iii) in Theorem 3.2 is valid: one has $\gamma = 1$ and $\partial G_\gamma(\xi) = \partial C_+(\xi)$.

Proof. The vector $r_\Lambda(z \mid X_{\bar{\Lambda}})$ [see (2.3)] is connected with the vector

$$f_\Lambda(z \mid X_{\bar{\Lambda}}) = \left\{\chi_\Lambda(X_m) z^m e^{-\beta U(X_m) - \beta W(X_m, X_{\bar{\Lambda}})}\right\}_{m\geqslant 1} \tag{3.9}$$

by the linear transformation $\mathcal{P}_\Lambda : E^s_\xi(\Lambda) \to E^s_\xi(\Lambda)$ (see Refs. 1, 6, and 7)

$$r_\Lambda(z \mid X_{\bar\Lambda}) = \mathcal{P}_\Lambda f_\Lambda(z \mid X_{\bar\Lambda})$$

$$\mathcal{P}_\Lambda = e^{\hat d_\Lambda}, \qquad (\hat d_\Lambda \varphi)(X_m) = \int_\Lambda dy\, \varphi_{m+1}(X_m \cup y) \tag{3.10}$$

$$\|\mathcal{P}_\Lambda\|_{E_\xi(\Lambda)} = \|\mathcal{P}_\Lambda^{-1}\|_{E_\xi(\Lambda)} = e^{\xi|\Lambda|}$$

By the properties of the pair potential, $f_\Lambda(z \mid X_{\bar\Lambda}) \in E^s_\xi(\Lambda)$ for $|z| \leqslant \xi$ and $f_\Lambda(z \mid X_{\bar\Lambda}) \notin E^s_\xi(\Lambda)$ for $|z| > \xi$; see (3.9). Therefore from (3.10) and the closed graph theorem one gets the same for the vector $r_\Lambda(z \mid X_{\bar\Lambda}) = \mathcal{P}_\Lambda f_\Lambda(z \mid X_{\bar\Lambda})$. ■

Superstable Interactions

For the superstable interaction one has[2]

$$U(X_n) \geqslant -n\left(B - \frac{n}{|\Lambda|} A\right), \qquad A > 0, \qquad B \geqslant 0, \qquad X_n \in \Lambda^n \tag{3.11}$$

In general, the condition (3.11) does not imply the boundedness of the KS operator but it gives $r_\Lambda(z \mid X_{\bar\Lambda}) \in E^s_\xi(\Lambda)$ for arbitrary $z \in \mathbb{C}$; see (2.5). Thus, the condition (iii) in Theorem 3.2 is now false. Nevertheless, we cannot assert that with increasing the repulsive part of the interaction [compare (2.4) and (3.11)] the residual spectrum for KR or KS operators disappears.

We can prove this property only for two classes of superstable interactions when one of the following additional conditions on the pair-potential repulsive part is imposed: (a) either the potential has a hard core singularity: $\Phi(x) = +\infty$ for $|x| < c$; (b) or $\Phi(x) \geqslant 0$ and $\Phi(x) > 0$ for some neighborhood of the origin (compare Theorem 3.4). Each of these conditions implies the superstability and, together with regularity (2.13), provides the boundedness of the KS operator (2.8) in the space E_ξ.

Theorem 3.5. Let $X_{\bar\Lambda} \in \Omega'_{\bar\Lambda}$ and the pair potential $\Phi(x)$ be regular and satisfy one of the conditions (a) or (b). Then the KS operator (2.8) with $D(K_\Lambda) = E^s_\xi(\Lambda)$ has only the point spectrum:

$$\sigma(K_\Lambda) = \sigma_p(K_\Lambda) = N(\Xi) \tag{3.12}$$

Proof. Using the transformation $\mathcal{P}_\Lambda$ (3.10) one can bring the operator $K_\Lambda \upharpoonright E^s_\xi(\Lambda)$ to a canonical form (see Refs. 1 and 6):

$$(\mathcal{P}_\Lambda^{-1} K_\Lambda \mathcal{P}_\Lambda \varphi)(x_1) = -\exp[-\beta W(x_1, X_{\bar\Lambda})]\chi_\Lambda(x_1) \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} dY_n\, \varphi_n(Y_n)$$

$$(\mathcal{P}_\Lambda^{-1} K_\Lambda \mathcal{P}_\Lambda \varphi)(X_m) = \exp[-\beta W(x_1, X_{\bar\Lambda}) - \beta W(x_1, X_m \backslash x_1)]$$
$$\times \chi_\Lambda(x_1) \varphi_{m-1}(X_m \backslash x_1), \qquad m \geqslant 2 \tag{3.13}$$

As $\mathcal{P}_\Lambda$ preserves the symmetry and the similarity transformation $\mathcal{P}_\Lambda^{-1} K_\Lambda \mathcal{P}_\Lambda = \tilde{K}_\Lambda$ does not change the spectrum, we reduce the problem to the determination of $\sigma(\tilde{K}_\Lambda)$ for $D(\tilde{K}_\Lambda) = E_\xi^s(\Lambda)$. Let

$$\varphi^{(n)} = -\sum_{l=1}^{\infty} \frac{1}{l!} \int_{\Lambda^l} dY_l (\tilde{K}_\Lambda^n \varphi)(Y_l), \qquad n \geqslant 0$$

Then straightforward calculations show that the operator $\tilde{K}_\Lambda^n$ has the form

$$\tilde{K}_\Lambda^n = \tilde{F}_n + \tilde{H}_n \tag{3.14}$$

$$\begin{aligned}
\tilde{F}_n \varphi = \big\{ & \chi_\Lambda(x_1) \exp\big[-\beta W(x_1, X_{\bar{\Lambda}})\big] \varphi^{(n-1)}, \\
& \chi_\Lambda(X_2) \exp\big[-\beta U(X_2) - \beta W(X_2, X_{\bar{\Lambda}})\big] \varphi^{(n-2)}, \ldots, \\
& \chi_\Lambda(X_n) \exp\big[-\beta U(X_n) - \beta W(X_n, X_{\bar{\Lambda}})\big] \varphi^{(0)}, 0, \ldots, 0, \ldots \big\} \\
\tilde{H}_n \varphi = \big\{ & 0, \ldots, 0, \chi_\Lambda(X_{n+1}) \exp\big[-\beta U(X_{n+1}) + \beta U(X_{n+1} \backslash X_n) \\
& \qquad - \beta W(X_{n+1}, X_{\bar{\Lambda}})\big] \varphi_1(X_{n+1} \backslash X_n), \ldots, \\
& \chi_\Lambda(X_{n+k}) \exp\big[-\beta U(X_{n+k}) + \beta U(X_{n+k} \backslash X_n) \\
& \qquad - \beta W(X_{n+k}, X_{\bar{\Lambda}})\big] \varphi_k(X_{n+k} \backslash X_n), \ldots \big\}
\end{aligned} \tag{3.15}$$

If the pair potential satisfies (a), then for the container Λ there exists a number $n_\Lambda(c)$ such that $U(X_n) = +\infty$ for $n > n_\Lambda(c)$. Hence, for $n > n_\Lambda(c)$ one has $\tilde{H}_n = 0$ [see (3.15)], and $\tilde{K}_\Lambda^n$ is equal to the finite rank operator $\tilde{F}_n$ [see (3.14)]. Therefore, $\sigma(\tilde{K}_\Lambda^n) = \sigma_p(\tilde{K}_\Lambda^n)$ and then $\sigma(K_\Lambda) = \sigma_p(K_\Lambda)$. Using (3.13) one can verify, (see Refs. 1 and 4), that for the operator $K_\Lambda \upharpoonright E_\xi^s(\Lambda)$ the set of eigenvalues coincides with $N(\Xi)$; here superstability implies that the eigenvectors $\{r_\Lambda(\lambda_i^{-1} \mid X_{\bar{\Lambda}})\}_{i \geqslant 1} \in E_\xi^s(\Lambda)$ for $\lambda_i \in N(\Xi)$.

Let now the pair potential satisfy (b). Then for any $\epsilon > 0$ there exists a number $n(\epsilon)$ such that for $n > n(\epsilon)$ one has [see (3.11) and (3.15)]

$$\left(\|\tilde{H}_n\|_{E_\xi^s(\Lambda)}\right)^{1/n} \leqslant \xi^{-1} e^{-n\beta A/|\Lambda|} < \epsilon \tag{3.16}$$

Consider the operator $(\lambda^n I - \tilde{K}_\Lambda^n)$ represented in the form

$$\lambda^n I - \tilde{K}_\Lambda^n = \left[\lambda^n I - \tilde{F}_n (I - \lambda^{-n} \tilde{H}_n)^{-1}\right](I - \lambda^{-n} \tilde{H}_n) \tag{3.17}$$

From the estimate (3.16) it follows that for $n > n(\epsilon)$ the spectrum of the operator $\tilde{K}_\Lambda^n \upharpoonright E_\xi^s(\Lambda)$ in the region $S_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| > \epsilon\}$ consists only of n eigenvalues. Let $\omega = \exp(2\pi i/n)$; then

$$\lambda^n I - \tilde{K}_\Lambda^n = (\lambda I - \tilde{K}_\Lambda)(\omega \lambda I - \tilde{K}_\Lambda) \cdots (\omega^{n-1} \lambda I - \tilde{K}_\Lambda) \tag{3.18}$$

The invertibility of $(\lambda^n I - \tilde{K}_\Lambda^n)$ implies the same for the operator $(\lambda I - \tilde{K}_\Lambda)$, hence one gets

$$[\sigma(\tilde{K}_\Lambda)]^n \cap S_\epsilon \subset \sigma(\tilde{K}_\Lambda^n) \cap S_\epsilon = \sigma_p(\tilde{K}_\Lambda^n) \cap S_\epsilon \qquad (3.19)$$

Therefore, the spectrum of the operator $\tilde{K}_\Lambda \upharpoonright E_\xi^s(\Lambda)$ in the region S_ϵ consists of a finite number of isolated points: $\sigma(\tilde{K}_\Lambda) \cap S_\epsilon \subset \sigma_p(\tilde{K}_\Lambda)$. Since the value of $\epsilon > 0$ may be arbitrarily small [see (3.16)], one gets $\sigma(\tilde{K}_\Lambda) = \sigma_p(\tilde{K}_\Lambda)$ and consequently $\sigma(K_\Lambda) = \sigma_p(K_\Lambda)$. The proof of the second part of equality (3.12) is the same as in the case (a). ■

Remark 3.5. From the proof one deduces that the spectral properties of the KS operator $K_\Lambda \upharpoonright E_\xi^s(\Lambda)$ are a consequence of its topological properties. Namely, in the case (a) the KS operator is potentially compact in the sense that K_Λ^n is compact for $n > n_\Lambda(c)$: $K_\Lambda^n = \mathcal{P}_\Lambda \tilde{F}_n \mathcal{P}_\Lambda^{-1}$; in the case (b) the KS operator is quasipotentially compact in the sense that for any $\epsilon > 0$ there exists a number $n(\epsilon)$ and a compact operator $F_{n(\epsilon)} = \mathcal{P}_\Lambda \tilde{F}_{n(\epsilon)} \mathcal{P}_\Lambda^{-1}$ such that $\|K_\Lambda^{n(\epsilon)} - F_{n(\epsilon)}\|_{E_\xi} < \epsilon^{n(\epsilon)}$ [see (3.16) and Ref. 8, Section X.5].

These properties can be expressed more precisely by the Fredholm-operator theory. We recall the reader (see, e.g., Ref. 9) that the operator $K_\Lambda(\lambda) = \lambda I - K_\Lambda$ is Fredholm at the point $\lambda \in \mathbb{C}$ [we denote $K_\Lambda(\lambda) \in \mathcal{F}(E_\xi^s(\Lambda))$] if it can be represented in the form

$$K_\Lambda(\lambda) = U_\lambda + T_\lambda \qquad (3.20)$$

Here $U_\lambda : E_\xi^s(\Lambda) \to E_\xi^s(\Lambda)$ is invertible and T_λ is compact. Then the number

$$\rho(K_\Lambda) = \inf_{\lambda \in \mathbb{C}} \{|\lambda| : K_\Lambda(\lambda) \in \mathcal{F}(E_\xi^s(\Lambda))\}$$

is called the Fredholm radius of the operator K_Λ with $D(K_\Lambda) = E_\xi^s(\Lambda)$. It is clear that $\rho(K_\Lambda) \leqslant r(K_\Lambda)$, where $r(K_\Lambda)$ is the spectral radius of K_Λ. Moreover, we show the following theorem.

Theorem 3.6. Let the assumptions of Theorem 3.5 be valid; then $\rho(K_\Lambda) = 0$.

Proof. The operator $K_\Lambda \upharpoonright E_\xi^s(\Lambda)$ is proved to have $\sigma(K_\Lambda) = \sigma_p(K_\Lambda)$ with the accumulation point at most at $\lambda = 0$; see Theorem 3.5. Then for any $\lambda \in \mathbb{C}$ there is a number n large enough such that we simultaneously have $(\|\tilde{H}_n\|_{E_\xi^s(\Lambda)})^{1/n} < |\lambda|\exp(-2\xi|\Lambda|/n)$ [see (3.16)] and $\{\lambda_k = \omega^k \lambda\}_{n=1}^{n-1} \not\subset \sigma(K_\Lambda)$, where $\omega = \exp(2\pi i/n)$. Therefore, from (3.14) and (3.18) it follows that $K_\Lambda(\lambda)$ is a Fredholm operator [see (3.10) and (3.20)]:

$$K_\Lambda(\lambda) = (\lambda^n I - H_n)V_n^{-1} - F_n V_n^{-1} \qquad (3.21)$$

here

$$V_n = (\omega\lambda I - K_\Lambda)(\omega^2\lambda I - K_\Lambda) \cdots (\omega^{n-1}\lambda I - K_\Lambda)$$

$$F_n = \mathcal{P}_\Lambda \tilde{F}_n \mathcal{P}_\Lambda^{-1}, \qquad H_n = \mathcal{P}_\Lambda \tilde{H}_n \mathcal{P}_\Lambda^{-1}$$

If one takes into account that the representation (3.21) is obtained for arbitrary $\lambda \neq 0$, then $\rho(K_\Lambda) = 0$. ■

Remark 3.6. In the Fredholm region of the operator K_Λ, here for $\lambda \in \mathbb{C}\backslash\{0\}$, the resolvent $R_\lambda(K_\Lambda)$ is known to be a finite-meromorphic function, see Refs. 8, 9.

4. DISCUSSION

The spectral properties of the KS operator in a container with empty boundary conditions (continuous or lattice systems) were considered for the first time by Pastur [6] under a restriction condition on the operator domain. It was supposed that $D(K_\Lambda) = \mathcal{D}_s$, where $\mathcal{D}_s \subset \bigcup_{\xi>0} E_\xi^s(\Lambda)$ is an invariant subspace of the operator K_Λ. Then the spectrum $\sigma(K_\Lambda \restriction \mathcal{D}_s)$ was shown to be pointlike. It coincides with the set $N(\Xi)$ and the KS equation has some properties similar to those of the Fredholm equations. Reformulation of Ref. 6 to the case of external fields is the subject of Ref. 10. In Ref. 11 there is an interesting attempt to determine the structure of the KS operator spectrum in infinite volume.

Note here that nonempty boundary conditions are equivalent to external fields [see Ref. 1 and (2.8), (2.9)] and have no influence on the spectral properties, defining only the position of the spectrum points on the plane $\mathbb{C}$.

Finally, the spectral properties of some "modified" KS operator in a container with empty boundary conditions for finite range hard core pair potentials were considered in Ref. 12. Some power of such a KS operator was shown to be a compact operator. Hence, the "modified" KS operator has a point spectrum (its explicit form has not been determined).

In the present paper the "ordinary" KS operator is shown to have the same property; see Theorem 3.5. Moreover, for superstable interactions, satisfying Theorem 3.5, the point spectrum and its structure are a consequence of the following general property: the KS equation is of the Fredholm type and the KS operator has a Fredholm radius equal to zero. These results are generalizations of those of Refs. 10 and 12 and partially of Ref. 6. We have found also that the decrease of the repulsive part of a pair potential (stable potentials) leads to the appearance of a residual spectrum for the KR operator (Theorem 3.3) and KS operator (Theorem 3.1), while for the KS operator for a stable nonzero potential we know only that $\sigma_p(K_\Lambda) \cup \tilde{\sigma}_p(K_\Lambda) = N(\Xi)$, $D(K_\Lambda) = E_\xi(\Lambda)$; see Theorem 3.2.

It should be noted here that, similarly to Theorems 3.5 and 3.6, spectral properties of the KS operator in the space $\mathfrak{D}_s$ are probably defined by its topological peculiarities if one endows $\mathfrak{D}_s$ with a natural topology; see Remark 3.2 in Ref. 13. We shall return to this problem in a subsequent paper.

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